Scale-free First Passage Percolation

[Scholars: Hansheng Liu, Shuzhen Zhang, Haotian Ju, Linzhe Teng] [Team Leader: Daecheol Kim, Aditya Gopalan]
[Faculty Advisor: Partha S. Dey]
University of Illinois Urbana–Champaign

Introduction

Consider rumor spreading as a percolation process.

Want to capture:

- How fast rumors spread (Determine the order of the speed)
- Effect of influencers (e.g. from a scale-free overlay network)
- Effect of network/physical distance on "communication rates" between people

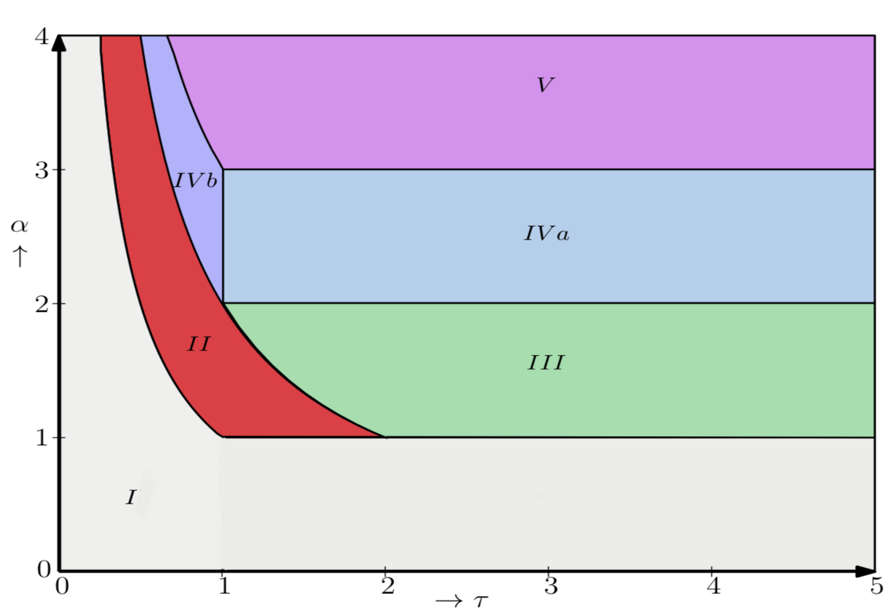
Model

- Complete Graph on vertex set $[-N, N]$
- α : distance cost exponent.
- γ : scale-free parameter.
- Node weight W_i given by $\mathbb{P}(W_i > x) = x^{-\gamma}$
- Edge cost of (i, j) is defined by:

$$|i-j|^{\alpha} \cdot \frac{\text{Exp}(1)}{W_i W_j}.$$

Goal

1. Simulate the first passage time $T_N := T(0, N)$ and visualize the phase transitions of it.



Phase diagram of T_N.

- I: Arbitrarily Small Growth: $O(1)$
- II: LogLog Growth: $O(\log\log N)$
- III: PolyLog Growth: $O((\log N)^{1/\log_2 \alpha + o(1)})$
- IV_a: Sublinear Growth: $O(N^{\alpha-2})$
- IV_b: Sublinear Growth: $O(N^{\alpha-2/\gamma})$
- V: Linear Growth: $O(N)$

2. Discover the path structure of each region.

Result 1: Convergence of average of T_N

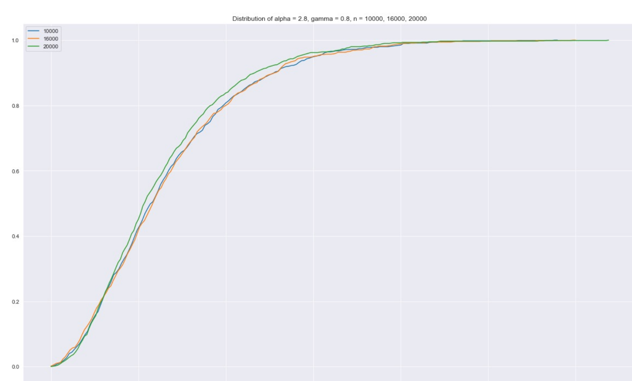
	$\alpha = 0.5$	$\alpha = 2.2$	$\alpha = 2.8$	$\alpha = 3.2$	$\alpha = 3.8$
$N = 8000$	7×10^{-5}	0.106	0.0719	0.0154	0.0135
$N = 16000$	4.75×10^{-5}	0.101	0.0681	0.0145	0.0131
$N = 20000$	3.82×10^{-5}	0.109	0.0643	0.0147	0.0128

Table 1: Average of T_N when $\gamma = 0.8$.

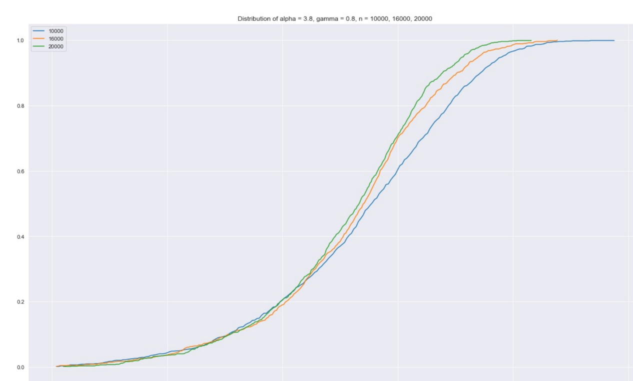
	$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 2.1$	$\alpha = 2.5$	$\alpha = 2.9$	$\alpha = 3.1$	$\alpha = 3.5$
$N = 8000$	0.016	1.50	4.69	0.705	0.106	0.0799	0.1591
$N = 16000$	0.013	1.57	4.51	0.722	0.104	0.0782	0.1592
$N = 20000$	0.012	1.62	5.48	0.733	0.103	0.0774	0.1590

Table 2: Average of T_N after scaling when $\gamma = 3$.

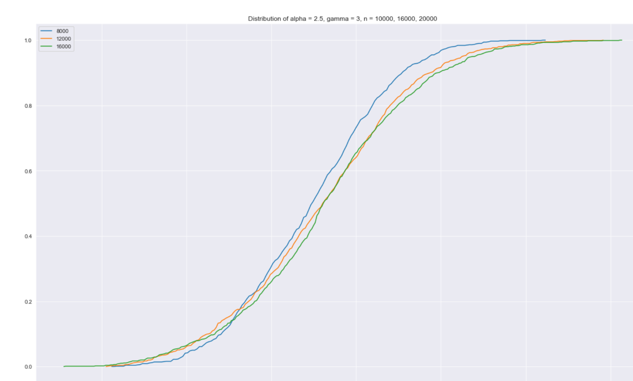
Result 2: Distributional Convergence of T_N



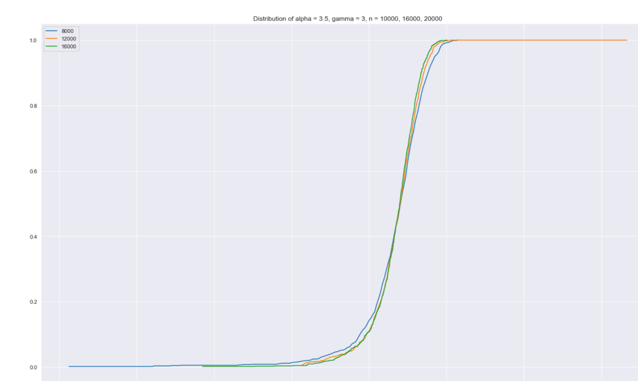
CDF of scaled T_N when $\alpha = 2.8, \gamma = 0.8$ (Region IV_b)



CDF of scaled T_N when $\alpha = 3.8, \gamma = 0.8$ (Region V)



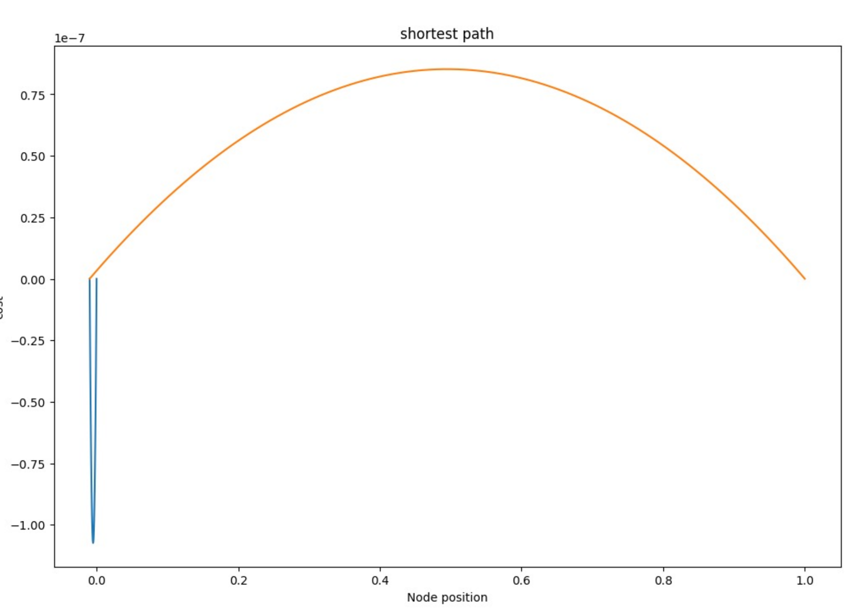
CDF of scaled T_N when $\alpha = 2.5, \gamma = 3$ (Region IV_a)



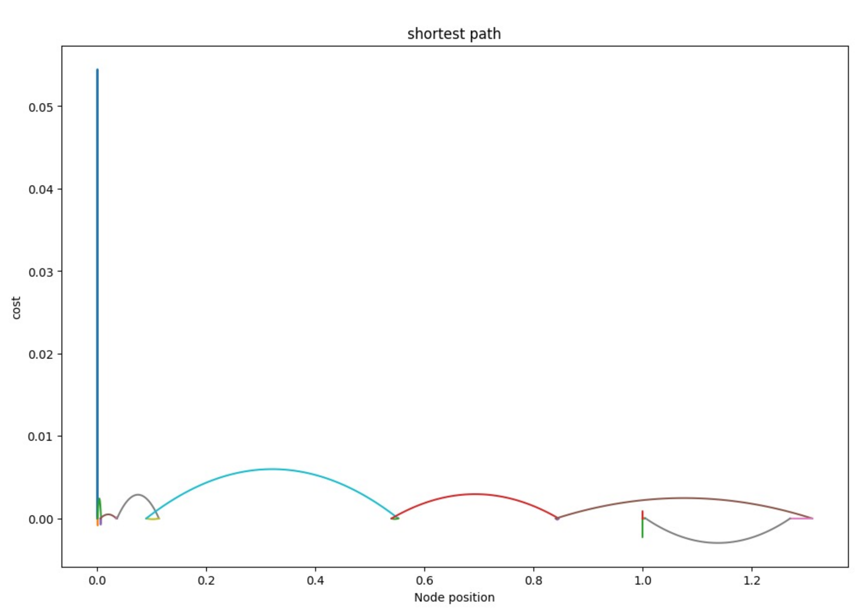
CDF of scaled T_N when $\alpha = 3.5, \gamma = 3$ (Region V)

Result 3: Path Structure

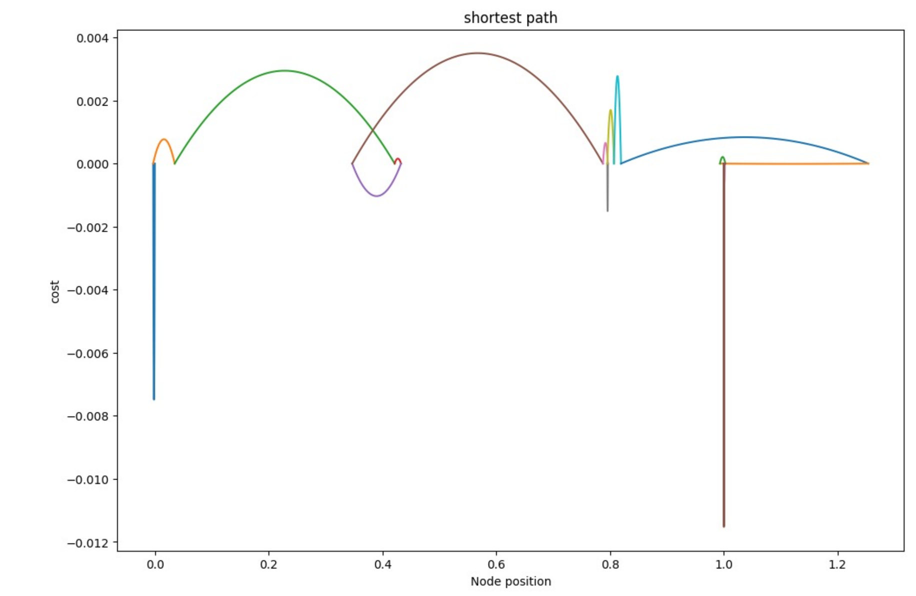
- Region I: There is only a few jumps.
- Region II & III: There are several large jumps.
- Region IV: There is exactly one large jumps with several small jumps.
- Region V: There are several small jumps with no significant large jumps.



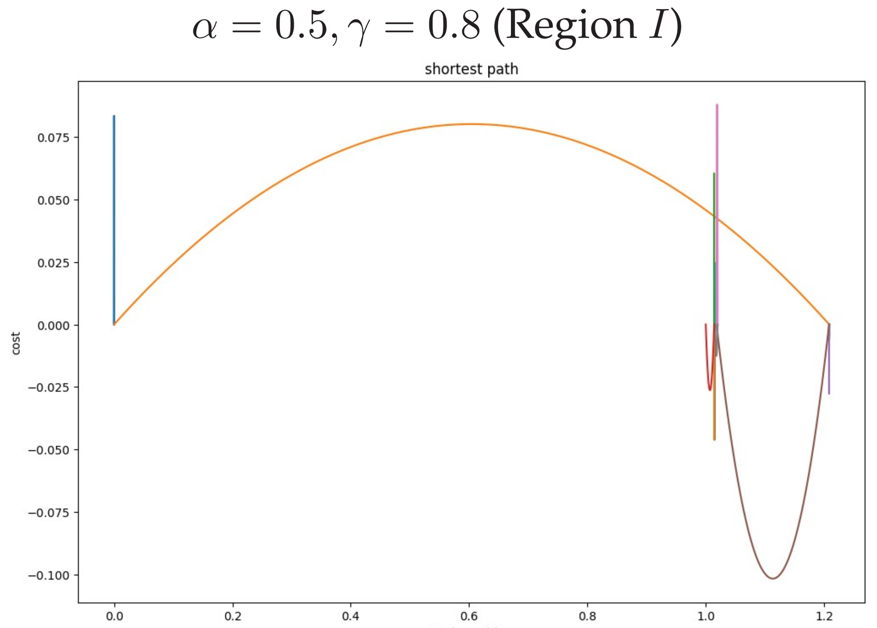
$\alpha = 0.5, \gamma = 0.8$ (Region I)



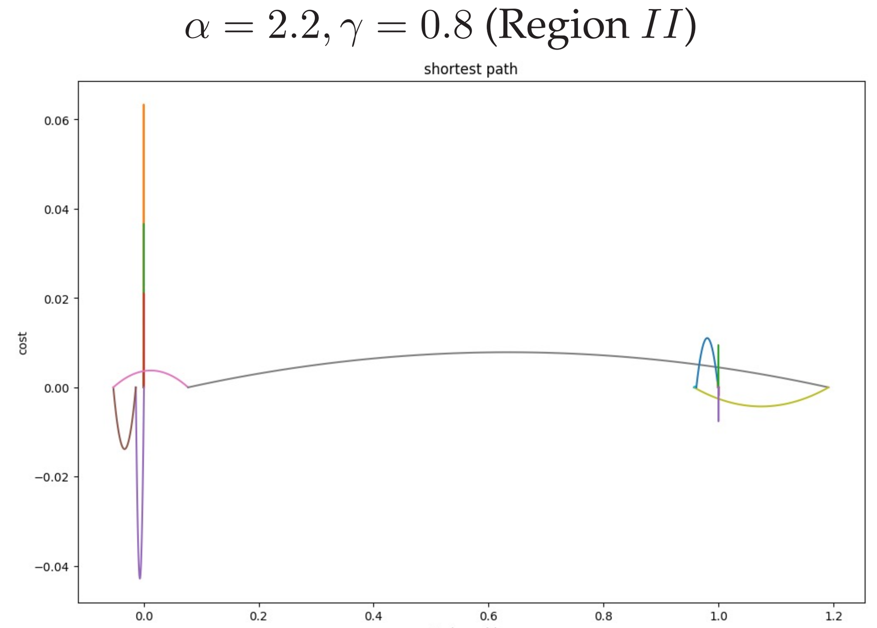
$\alpha = 2.2, \gamma = 0.8$ (Region II)



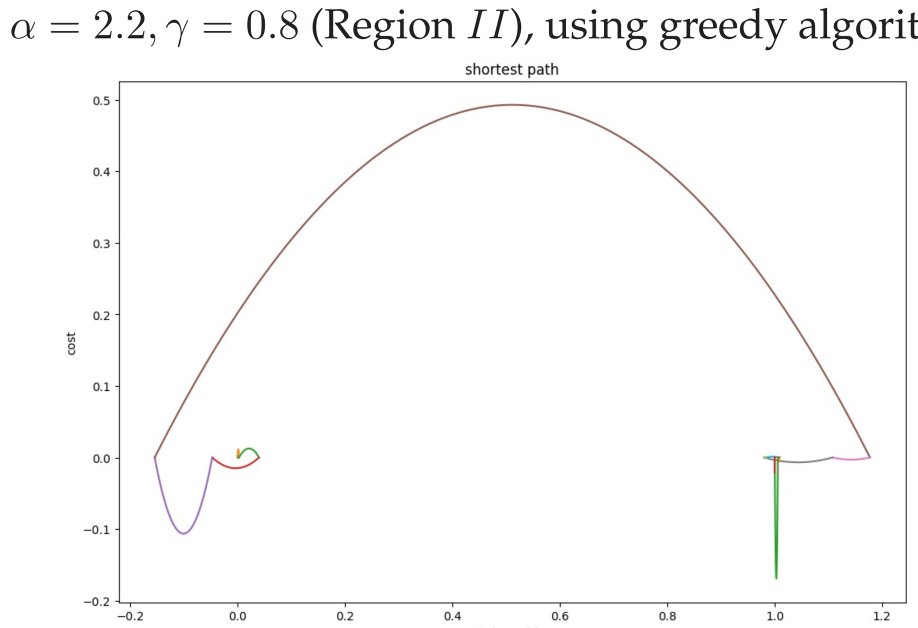
$\alpha = 2.2, \gamma = 0.8$ (Region II), using greedy algorithm



$\alpha = 1.5, \gamma = 3.0$ (Region III)



$\alpha = 2.8, \gamma = 0.8$ (Region IV_b)



$\alpha = 3.0, \gamma = 0.8$ (Region IV_b), using greedy algorithm

Greedy Algorithm

In some regions, an upper bound of T_N can be obtained easily using a greedy algorithm:

II. Pick $\theta > 1$. Find the node with the largest weight among nodes from the starting node to $\frac{N}{\theta}$. Recursively find maximum weight nodes from starting node to $\frac{N}{\theta^2}, \frac{N}{\theta^3}, \ldots$ until $\frac{N}{\theta^k}$ is close enough to the starting node for some k. Do the same thing from the right side with the target node. Connect all these nodes to construct the path.

III. Pick $0 < \theta < 1$. Find the shortest edge from a random node n_1 in $[0, N^{\theta}]$ to another random node n_2 in $[N - N^{\theta}, N]$. Recursively apply this algorithm to the sets $[0, n_1]$, and $[n_2, N]$.

IV_a. Pick $\theta > 1$. Find the shortest edge from a random node n_1 in $[0, \frac{N}{\theta}]$ to another random node n_2 in $[N - \frac{N}{\theta}, N]$. Recursively apply this algorithm to the sets $[0, n_1]$, and $[n_2, N]$.

IV_b. Pick $0 < \theta < 1$. Find the node with largest weight in $[0, N^{\theta}]$. Recursively find nodes from starting node to $N^{\theta^2}, N^{\theta^3}, \ldots$ until N^{θ^k} is close enough to the starting node for some k. Do the same thing from the right side with target node. Connect all these nodes to construct the path.

Conjectures

1. First passage time for each region has different estimated order based on our simulations.
2. Each region exhibits unique path structures.

Future Goals

1. Increase N to observe convergence effects.
2. Investigate the greedy algorithm for consistency with the original algorithm's results.
3. Experiment with additional algorithms to further reduce runtime, particularly for large N values.

References

1. Chatterjee, S., & Dey, P. S. (2016). Multiple Phase Transitions in Long-Range First-Passage Percolation on Square Lattices. *CPAM*, 69(2), 203–256.
2. Deijfen, M., Van der Hofstad, R., & Hooghiemstra, G. (2013). Scale-free percolation. *AIHP*, Vol. 49, No. 3, pp. 817–838.

IGL Poster Session Fall 2023. Support for this project was provided by the Illinois Geometry Lab and the Department of Mathematics at the University of Illinois at Urbana-Champaign.